Temperaments for isomorphic keyboards

Jan Behrens

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Abstract

These notes present and compare some previously developed tuning systems suitable for isomorphic keyboards. One of them utilizes the golden ratio ($\phi = \frac{\sqrt{5}+1}{2}$) as the ratio between the logarithmic frequency ratios of whole tones and (diatonic) semitones. It is constructed by demanding a an exact ratio of $2:1$ for the octave and further split the octave into 5 equal whole tones and 2 equal semitones, for which the ratio between the logarithm of the frequency ratios of whole tones ($\log_2 w$) and the logarithm of the frequency ratios of semitones ($\log_2 s$) is the golden ratio ($\log_2 w : \log_2 s = \phi$). As can be shown, this choice provides almost optimal approximation of all frequency ratios involved for major and minor chords ($\frac{5}{4} \times \frac{6}{5} = \frac{3}{2}$) as well as their septimal variants ($\frac{9}{7} \times \frac{7}{6} = \frac{3}{2}$) when compared to just intonation.

Furthermore, a rounded tuning with $\boldsymbol{w = 192.6\,\mathrm{ct}}$ and $\boldsymbol{s = 118.5\,\mathrm{ct}}$ is suggested, which approximates both the golden ratio and the optimization of septimal and non-septimal chords. Errors for all discussed tunings compared to just intonation are provided in a tabular overview.

1 Introduction

1.1 Whole tones and semitones

The diatonic scale splits an octave (frequency ratio $^2/_1$) into 5 bigger steps, called *"whole tones"*, and 2 smaller steps, called *"semitones"*. Piano keyboards provide white keys with notes named A, B, C, D, E, F, G, A, . . . , also called *"natural notes"*, where the steps between the natural notes B and C and the steps between the natural notes E and F are semitones, while all other steps between the notes of two adjacent natural notes are whole tones. Additional black keys are provided to further divide all whole tone steps into two semitone steps by introducing altered notes such as C♯ or D♭, where C → C♯ → D, or C → D♭ → D would be semitone steps.

Depending on the tuning of the keyboard, it may be necessary to distinguish between *diatonic* semitone steps and *chromatic* semitone steps. In case of a diatonic semitone step, the note letter changes (e. g. C♯ → D, or C → D♭), while for chromatic semitone steps, only the accidentals (♯, ♭) change (e. g. C → C♯, or D♭ → D).

Traditional piano layouts only provide up to one black key in between two white keys, e. g. there is only key for C♯ and D♭. When diatonic semitones and chromatic semitones are tuned to be slightly different steps than chromatic semitones, it matters whether you play, for example, a *C♯* or *D♭*. In this case diatonic semitone steps are usually larger than chromatic ones, which results in the following ordering of steps, from largest to smallest step size:

- whole tone
- diatonic semitone
- chromatic semitone

1.2 Isomorphic layout

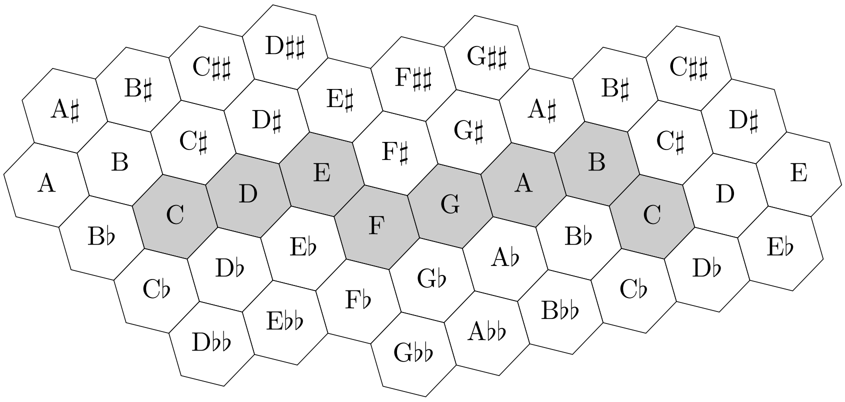

Figure 1: Isomorphic keyboard

An isomorphic keyboard maps the same musical intervals (see subsection 1.3) to the same shape on the keyboard, invariant to transposing. Using hexagonal keys, each key (except keys at the outer border) has 6 adjacent keys. Assuming that all whole tones have the same frequency ratio w and all diatonic semitones have the same frequency ratio s, each of those 6 directions can be made to represent one of the following steps:

- whole tone up (w)
- diatonic semitone up (s)
- chromatic semitone down ($c^{-1} = w^{-1}s$)
- whole tone down (w^{-1})
- diatonic semitone down (s^{-1})
- chromatic semitone up ($c = ws^{-1}$)

Note that chromatic semitones can be described by a combination of whole tones and diatonic semitones, i. e. $c = \frac{w}{s}$. Any interval on the keyboard can be described by two integers i, j, where i is the number of whole tones up and j is the number of diatonic semitones up. The frequency ratio between the notes of the corresponding keys then calculates as $w^i s^j$.

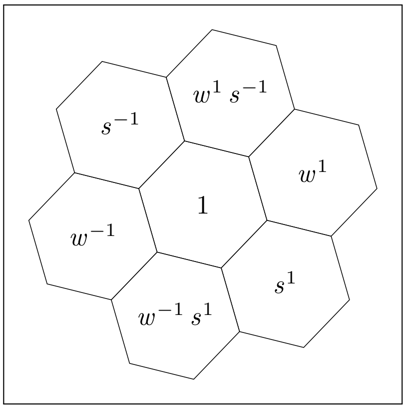

Figure 2: Relative frequencies on the isomorphic keyboard

1.3 Musical intervals and overtones

Differences in pitch (i. e. frequency ratios) between two notes are called *"intervals"*. These frequency relationships between notes would ideally be exact rational numbers, such that specific overtones (harmonics of non-sine waveforms) coincide. When exact rational numbers are used, this is called *"just intonation"*. It is possible, however, to create approximations by combining whole tone steps and semitone steps. The following table gives an overview of the most relevant intervals resulting from i whole tones and j diatonic semitones.

By calculating $1 + i + j$, we obtain a name for the corresponding interval (1 = "prime", 2 = "second", ...). Given a starting note, the sum of i and j also determines the name of the resulting note (possibly with additional sharps ♯ or flats ♭) when performing the steps. The notes in the following table are provided as an example only, using C as an arbitrary starting point.

Table 1: Intervals built from i whole tones and j diatonic semitones

interval	just int.	i	j	note example	$1+i+j$
prime	$1/1$	0	0	C	1
minor second	$16/15$	0	1	D♭	2
major second	$10/9$ or $9/8$	1	0	D	2
minor third	$6/5$	1	1	E♭	3
major third	$5/4$	2	0	E	3
fourth	$4/3$	2	1	F	4

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Table 1: Intervals built from i whole tones and j diatonic semitones (Continued)

interval	just int.	i	j	note example	$1+i+j$
fifth	$3/2$	3	1	G	5
minor sixth	$8/5$	3	2	A♭	6
major sixth	$5/3$	4	1	A	6
minor seventh	$9/5$	4	2	B♭	7
major seventh	$15/8$	5	1	B	7
octave	$2/1$	5	2	C	8

The specific choice of w and s results in different errors between the ideal fractional ratio of just intonation and the resulting approximation $w^i s^j$. For the tunings discussed in this document, an overview is given later in section 3.

Frequency ratios with a small numerator and denominator (after simplifying the fraction) are most relevant as more overtones will coincide. In the table above, fractions containing the prime factor 7 (such as $7/6$) are left out yet, but they are revisited in subsection 1.5 and later sections.

1.4 Major and minor chords

In just intonation, we can split a fifth ($3/2$) into two thirds, of which one third is a major third ($5/4$) and the other one is a minor third ($6/5$), because $5/4 \times 6/5 = 3/2$.

Depending on the ordering of the major and minor thirds, we obtain major (e. g. C E G) or minor (e. g. C E♭ G) chords. A musical tuning utilizing whole tone and semitone steps ideally creates only a small error in the frequency ratios when compared to just intonation.

1.5 Septimal thirds

An alternative way to split a fifth ($3/2$) into two rational numbers is to use $9/7$ for the major third and $7/6$ for the minor third, because $9/7 \times 7/6 = 3/2$. In the following we call these thirds *"septimal variants of thirds"* due to their role in splitting a fifth, even when we have to choose i and j such that $1 + i + j \neq 3$.

1.6 Logarithmic pitch differences in cents

The pitch difference of notes or musical intervals is often described using the logarithmic unit *"cent"* (ct) by taking the frequency ratio r and calculating $1200\,\mathrm{ct} \times \log_2 r$. Then 1200 cents describes an octave, while smaller values correspond to smaller intervals.

In twelve-tone equal temperament, where an octave is split into 12 equal semitone steps, 100 cents would be a semitone (diatonic and chromatic), and 200 cents a whole tone. Cent values for whole tones and semitones of the other tunings discussed in this document are given later in section 3, Table 5.

2 Construction of tunings

2.1 Octave requirement

Assuming that 5 whole tones and 2 diatonic semitones shall result in a perfect octave (frequency ratio $2/1$), we get the equation $w^5 s^2 = 2$. This means that s can be calculated in terms of w, or vice versa. Specifically:

$$s = \sqrt{\frac{2}{w^5}}$$

This results in one remaining degree of freedom, i. e. if we choose some value for w, then this results in specific values for s and c.

2.2 Quarter-comma meantone temperament

The well-known quarter-comma meantone temperament chooses $w = \sqrt{5/4}$ such that two whole tones result in a major third with no error ($w^2 = 5/4$). This minimizes the maximum absolute logarithmic error in the frequency ratios involved in major and minor chords when using two whole tones for major thirds and one whole and one diatonic semitone for minor thirds.

Using the equation $w^5 s^2 = 2$, we get $s = \sqrt{2 \cdot \sqrt{(4/5)^5}}$.

Note that using two whole tones minus one diatonic semitone results in a septimal variant of the minor third ($w^2\, s^{-1} \approx 7/6$) and one whole tone plus two diatonic semitones result in a septimal variant of the major third ($w^1\, s^2 \approx 9/7$), both with a non-optimal approximation, however. The traditional quarter-comma meantone temperament only optimizes the errors in the non-septimal major and minor chords by equally distributing the error between the minor third and the fifth.

As the septimal thirds are constructed with $1+i+j = 2$ and 4, respectively, we can refer to the septimal variants of the thirds also as *"augmented second"* (which is a major second plus a chromatic semitone) and *"diminished fourth"* (a fourth minus a chromatic semitone).

Table 2: Interval errors for chords in meantone temperament

interval	just int.	i	j	$w^i s^j$	error
prime	$1/1$	0	0	1	$1 = \pm 0.00\,\text{ct}$
sept. minor third (augmented second)	$7/6$	2	-1	$25\sqrt[4]{5}/32$	$75\sqrt[4]{5}/112 \approx +2.33\,\text{ct}$
minor third	$6/5$	1	1	$4\sqrt[4]{5}/5$	$2\sqrt[4]{5}/3 \approx -5.38\,\text{ct}$
major third	$5/4$	2	0	$5/4$	$1 = \pm 0.00\,\text{ct}$
sept. major third (diminished fourth)	$9/7$	1	2	$32/25$	$224/225 \approx -7.71\,\text{ct}$
fifth	$3/2$	3	1	$\sqrt[4]{5}$	$2\sqrt[4]{5}/3 \approx -5.38\,\text{ct}$

2.3 Modified meantone temperament by optimizing septimal thirds ("MT7")

In order to minimize the maximum absolute logarithmic error of major and minor chords both in their septimal and non-septimal form, it is possible to slightly modify the quarter-comma meantone temperament and choose $w = \sqrt[9]{({}^7\!/\!{}_3)^2 : 2}$, resulting in $s = \sqrt[9]{4 \cdot ({}^6\!/\!{}_7)^5}$. This tuning has been previously referred to as "meantone with pure 7:6".[1] In these notes, it will be briefly called *"MT7"*. It results in the following errors:

Table 3: Interval errors for chords in MT7 tuning

interval	just int.	i	j	$w^i s^j$	error
prime	$1/1$	0	0	1	$1 = \pm 0.00\,\mathrm{ct}$
sept. minor third (augmented second)	$7/6$	2	-1	$7/6$	$1 = \pm 0.00\,\mathrm{ct}$
minor third	$6/5$	1	1	$\sqrt[3]{12/7}$	$\sqrt[3]{125/126} \approx -4.60\,\mathrm{ct}$
major third	$5/4$	2	0	$\sqrt[9]{\frac{2401}{324}}$	$\sqrt[9]{\frac{157351936}{158203125}} \approx -1.04\,\mathrm{ct}$
sept. major third (diminished fourth)	$9/7$	1	2	$\sqrt[9]{\frac{53747712}{5764801}}$	$\sqrt[9]{\frac{57344}{59049}} \approx -5.64\,\mathrm{ct}$
fifth	$3/2$	3	1	$\sqrt[9]{112/3}$	$\sqrt[9]{\frac{57\,344}{59\,049}} \approx -5.64\,\mathrm{ct}$

2.4 Golden meantone

We observe that our previous choice of $w = \sqrt[9]{({}^7\!/\!{}_3)^2 : 2}$ and $s = \sqrt[9]{4 \cdot ({}^6\!/\!{}_7)^5}$ in subsection 2.3 approximately resembles the golden ratio $\phi = (\sqrt{5} + 1) : 2$ when we consider their logarithms (i. e. their values in cents). With $w \approx 192.638\,\mathrm{ct}$ and $s \approx 118.405\,\mathrm{ct}$, we get $\log_2 w : \log_2 s \approx \frac{192.638}{118.405} \approx 1.62694 \approx \phi = 1.61803$. The mismatch of the ratio of the logarithms compared to the golden ratio is less than 0.6 %.

Under the condition that $w^5 s^2 = 2$, it is possible to demand that the equation $\log_2 w : \log_2 s = \phi$ holds exactly and not just approximately. In that case,

$$w = 2^{\frac{4-\sqrt{5}}{11}}$$

and

$$s = 2^{\frac{5\sqrt{5}-9}{22}}.$$

This tuning system has been previously described as *"golden meantone"*.[2]

As it is an inherent property of the golden ratio that $(\phi + 1) : \phi = \phi$, we obtain a beautiful property regarding the whole tone w, the diatonic semitone s, and the chromatic semitone $c = \frac{w}{s}$:

$$\frac{\log_2 w}{\log_2 s} = \frac{\log_2 s}{\log_2 c} = \phi$$

[1] `https://scalelibrary.org/scales/mailing-lists/pure7-6mnt/`

[2] `https://en.xen.wiki/w/Golden_meantone`

In other words: Considering their values in cents (logarithmic frequency ratios), a whole tone relates to a diatonic semitone the same way as a diatonic semitone relates to a chromatic semitone.

We observe the following errors when comparing golden meantone with just intonation:

Table 4: Interval errors for chords in golden meantone

interval	just int.	i	j	$w^i s^j$	error
prime	$1/1$	0	0	1	$\pm 0.00\,\mathrm{ct}$
sept. minor third (augmented second)	$7/6$	2	-1	$2^{(25-9\sqrt{5}):22}$	$-0.94\,\mathrm{ct}$
minor third	$6/5$	1	1	$2^{(3\sqrt{5}-1):22}$	$-4.28\,\mathrm{ct}$
major third	$5/4$	2	0	$2^{2(4-\sqrt{5}):11}$	$-1.46\,\mathrm{ct}$
sept. major third (diminished fourth)	$9/7$	1	2	$2^{(4\sqrt{5}-5):11}$	$-4.80\,\mathrm{ct}$
fifth	$3/2$	3	1	$2^{(15-\sqrt{5}):22}$	$-5.74\,\mathrm{ct}$

The fifth has a slightly increased error of -5.74 cents compared to -5.38 cents in quarter-comma meantone temperament or -5.64 cents in the MT7 tuning. To the human ear, these errors are likely neglectable. For completeness, however, it should be noted that when resembling the harmonic seventh ($7/4$) using 5 whole tones, a slightly higher error of -6.68 cents occurs, while the MT7 tuning would only result in an error of -5.64 cents. All these errors, however, have a smaller absolute cent value than the error of the septimal major third in quarter-comma meantone temperament (which is -7.71 cents). In the last section 3, there is an overview of the different tunings and their errors for various intervals.

2.5 Rounded decimal values in cents

It should be noted that the tunings in subsection 2.3 (MT7 tuning) and subsection 2.4 (golden meantone) are almost identical and likely indistinguishable for most human listeners. A whole tone in MT7 is $\approx 192.638\,\mathrm{ct}$ while a whole tone in golden meantone is $\approx 192.429\,\mathrm{ct}$.

Rounding the values for whole tones and diatonic semitones to one decimal digit in the cent value under the condition that 5 whole tones and 2 diatonic semitones shall result in no error regarding the octave, and choosing rounding up or down in such a way that both tunings are approximated, we get

$$w = 192.6\,\mathrm{ct}$$

and

$$s = 118.5\,\mathrm{ct}$$

as a compromise between those tunings with easy to remember cent values, as $s = 118\frac{1}{2}\,\mathrm{ct}$ and w can be calculated from that ($w[\mathrm{ct}] = (1200 - 2 \times s[\mathrm{ct}]) : 5$). The chromatic semitone results in $c = 74.1\,\mathrm{ct}$.

3 Overview of steps and resulting errors

Table 5: Comparison of steps in each tuning

step	meantone	MT7	golden	rounded
whole tone (w)	193.16 ct	192.64 ct	192.43 ct	192.6 ct
diatonic semitone (s)	117.11 ct	118.41 ct	118.93 ct	118.5 ct
chromatic semitone (c)	76.05 ct	74.23 ct	73.50 ct	74.1 ct
enharmonic difference ($\frac{s}{c} = w^{-1}s^2$)	41.06 ct	44.17 ct	45.43 ct	44.4 ct

Table 6: Musical intervals with small numerators/denominators

interval	just int.	i	j	note example	$1+i+j$
prime	1/1	0	0	C	1
major second	10/9 or 9/8	1	0	D	2
sept. major second (diminished third)	8/7	0	2	E♭♭	3
sept. minor third (augmented second)	7/6	2	−1	D♯	2
minor third	6/5	1	1	E♭	3
major third	5/4	2	0	E	3
sept. major third (diminished fourth)	9/7	1	2	F♭	4
fourth	4/3	2	1	F	4
tritonus (augmented fourth)	7/5	3	0	F♯	4
tritonus compl. (diminished fifth)	10/7	2	2	G♭	5
fifth	3/2	3	1	G	5
minor sixth	8/5	3	2	A♭	6
major sixth	5/3	4	1	A	6
harmonic seventh (augmented sixth)	7/4	5	0	A♯	6
minor seventh	9/5	4	2	B♭	7
octave	2/1	5	2	C	8

Table 7: Errors of relevant intervals

interval	just	meantone	MT7	golden	rounded
prime	$1/1$	±0.00 ct	±0.00 ct	±0.00 ct	±0.00 ct
major second	$10/9$	+10.75 ct	+10.23 ct	+10.03 ct	+10.20 ct
	$9/8$	−10.75 ct	−11.27 ct	−11.48 ct	−11.31 ct
sept. major second (diminished third)	$8/7$	+3.04 ct	+5.64 ct	+6.68 ct	+5.83 ct
sept. minor third (augmented second)	$7/6$	+2.33 ct	±0.00 ct	−0.94 ct	−0.17 ct
minor third	$6/5$	−5.38 ct	−4.60 ct	−4.28 ct	−4.54 ct
major third	$5/4$	±0.00 ct	−1.04 ct	−1.46 ct	−1.11 ct
sept. major third (diminished fourth)	$9/7$	−7.71 ct	−5.64 ct	−4.80 ct	−5.48 ct
fourth	$4/3$	+5.38 ct	+5.64 ct	+5.74 ct	+5.66 ct
tritonus (augmented fourth)	$7/5$	−3.04 ct	−4.60 ct	−5.23 ct	−4.71 ct
tritonus compl. (diminished fifth)	$10/7$	+3.04 ct	+4.60 ct	+5.23 ct	+4.71 ct
fifth	$3/2$	−5.38 ct	−5.64 ct	−5.74 ct	−5.66 ct
minor sixth	$8/5$	±0.00 ct	+1.04 ct	+1.46 ct	+1.11 ct
major sixth	$5/3$	+5.38 ct	+4.60 ct	+4.28 ct	+4.54 ct
harmonic seventh (augmented sixth)	$7/4$	−3.04 ct	−5.64 ct	−6.68 ct	−5.83 ct
minor seventh	$9/5$	−10.75 ct	−10.23 ct	−10.03 ct	−10.20 ct
octave	$2/1$	±0.00 ct	±0.00 ct	±0.00 ct	±0.00 ct