

A two-dimensional musical scale

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Abstract

This document describes a tuning with associated nomenclature for isomorphic keyboards based on a 5-tone scale. Compared to just intonation, the tuning achieves interval errors smaller than 5.35 ct for many desired frequency ratios. On a two-dimensional isomorphic keyboard, relevant intervals do not lie close to a line but rather cover a two-dimensional area.

1 Definition of the scale and nomenclature

1.1 Base scale with 5 steps

The base scale consists of 5 steps (i. e. frequency ratios), consisting of 4 whole tone steps (w) and 1 semitone step (s), which result in an octave (such that $w^4 s = 2$). We define:

$$\begin{aligned} w &\stackrel{\text{def}}{=} \sqrt[17]{2^3 \cdot 3^2 \cdot 5^{-1}} = \sqrt[17]{\frac{72}{5}} \\ s &\stackrel{\text{def}}{=} \sqrt[17]{2^5 \cdot 3^{-8} \cdot 5^4} = \sqrt[17]{\frac{20\,000}{6\,561}} \\ c &\stackrel{\text{def}}{=} \sqrt[17]{2^{-2} \cdot 3^{10} \cdot 5^{-5}} = \sqrt[17]{\frac{59\,049}{12\,500}} \end{aligned}$$

where $c = \frac{w}{s}$ is called a chromatic step.

In the logarithmic unit “cents” (ct), where 1200 cents correspond to the frequency ratio $2^{1/1}$ of an octave, these calculate approximately to:

$$\begin{aligned} w &\approx 271.6 \text{ ct} \\ s &\approx 113.6 \text{ ct} \\ c &\approx 158.0 \text{ ct} \end{aligned}$$

Note that the above cent values were rounded such that $w^4 s = 2$ holds exactly ($4 \times 271.6 + 113.6 = 1200$).

1.2 Scale on hexagonal keyboards

On hexagonal keyboards, whole tone steps (w), semitone steps (s), and chromatic steps (c) can be mapped to three different adjacent keys, where the opposite direction corresponds to the respective inverse steps (w^{-1} , s^{-1} , and c^{-1}).

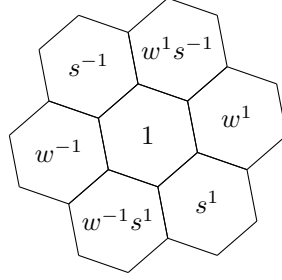


Figure 1: Steps on the isomorphic keyboard with their relative frequencies

This results in an isomorphic keyboard, where the same chord results in the same geometric shape, invariant under transposition.

As a chromatic step is equal to a whole tone up and a semitone down, each key on the keyboard can be located using two indices $i, j \in \mathbb{Z}$, where $i \in \mathbb{Z}$ is the number of whole tones up and $j \in \mathbb{Z}$ is the number of semitones up from a given starting point (in subsection 1.5, the note named γ_4 will be used as a starting point).

It is possible to rotate the hexagons such that an octave ($w^4s = 2$) is located on a horizontal line. Starting from an orientation in which whole tones are horizontal, the required rotation angle (also used in all figures) is:

$$\arctan \frac{\sqrt{3}}{9} \approx 10.89^\circ.$$

1.3 Five base notes per octave

The five base notes are called $\alpha, \beta, \gamma, \delta, \epsilon$. They can be indexed with an octave number $h \in \mathbb{Z}$, e.g. $\alpha_{-1}, \alpha_0, \alpha_1, \dots$. When two notes X_{h_1} and X_{h_2} with $X \in \{\alpha, \beta, \gamma, \delta, \epsilon\}$ and $h_1, h_2 \in \mathbb{Z}$ only differ in their octave numbers h_1 and h_2 , then their frequency ratio ($X_{h_1} : X_{h_2}$) is $2^{h_1-h_2}$, e.g. α_1 has a frequency twice as high as α_0 .

For all octave indices $h \in \mathbb{Z}$, the intervals from $\alpha_h \rightarrow \beta_h, \beta_h \rightarrow \gamma_h, \gamma_h \rightarrow \delta_h$, and $\delta_h \rightarrow \epsilon_h$ are whole tone steps (w) and the intervals from $\epsilon_h \rightarrow \alpha_{h+1}$ are semitone steps (s).

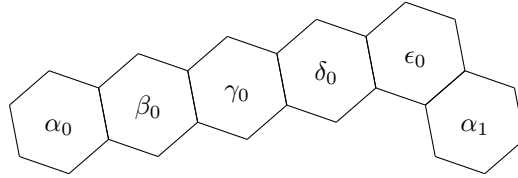


Figure 2: Base notes

1.4 Chromatic alteration

For each base note, there exist chromatic alterations by going $k \in \mathbb{Z}$ chromatic steps (c) up or down. This is denoted by arrows ($\uparrow$ and $\downarrow$) as superscripts, e.g.

$\gamma_0^\uparrow$, $\gamma_0^{\uparrow\uparrow}$, or $\gamma_0^\downarrow$. More than two chromatic steps up or down are denoted by an arrow followed by an Arabic number, e. g. $\gamma_0^{\uparrow 3}$ or $\gamma_0^{\downarrow 3}$. See Figure 3 for examples on a hexagonal isomorphic keyboard.

1.5 Frequency calculation

By the following definition, γ_4 is set to correspond to the frequency of D_4 in twelve-tone equal temperament (12 TET):

$$\gamma_4 \stackrel{\text{def}}{=} 2^{\frac{-7}{12}} \cdot 440 \text{ Hz} \approx 293.6648 \text{ Hz}.$$

We further define:

$$f(h, i, j, k) \stackrel{\text{def}}{=} \gamma_4 \cdot 2^{h-4} \cdot w^i \cdot s^j \cdot c^k.$$

For a given octave number $h \in \mathbb{Z}$ as well as a specified count of whole tone steps $i \in \{-2, -1, 0, 1, 2\}$ from γ_h (corresponding to $\alpha_h, \beta_h, \gamma_h, \delta_h, \epsilon_h$), and a chromatic step count $k \in \mathbb{Z}$, the absolute frequency is calculated as

$$f(h, i, 0, k) = \gamma_4 \cdot 2^{h-4} \cdot w^i \cdot c^k.$$

Alternatively, going $i \in \mathbb{Z}$ whole tone steps up and $j \in \mathbb{Z}$ semitone steps up from γ_4 , the frequency is calculated as

$$f(4, i, j, 0) = \gamma_4 \cdot w^i \cdot s^j.$$

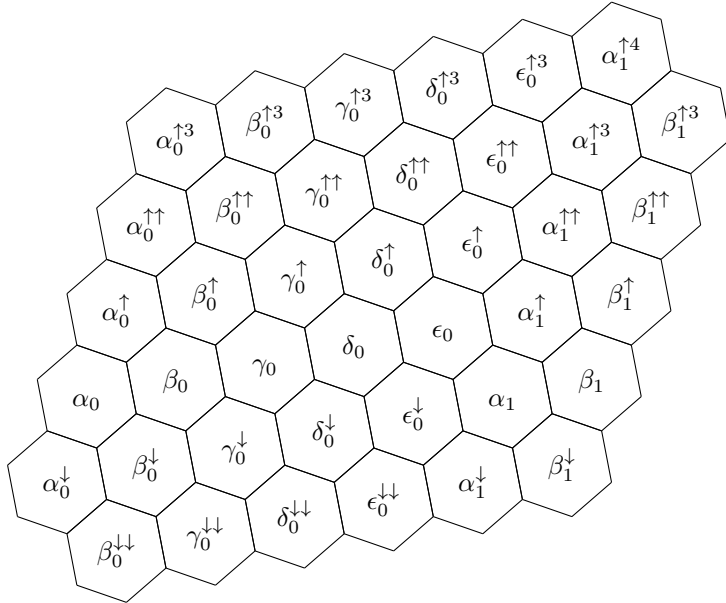


Figure 3: Base notes and chromatic alterations

2 Intervals and chords

2.1 Approximated intervals compared to just intonation

Using a combination of whole tones and semitones, it is possible to construct several musical intervals. In the following Table 1, frequency ratios along with their approximation by i whole tones and j semitones and the resulting error in cents are listed.

Table 1: Intervals built from i whole tones and j semitones

interval	just int.	i	j	note example	error
prime	$1/1$	0	0	γ_4	± 0.00 ct
minor second	$16/15$	0	1	$\delta_4^\downarrow$	+1.78 ct
narrow major second	$10/9$	-1	4	$\alpha_5^{\downarrow 3}$	± 0.00 ct
wide major second	$9/8$	2	-3	$\beta_4^{\uparrow 3}$	-1.18 ct
sept. major second	$8/7$	0	2	$\epsilon_4^{\downarrow \downarrow}$	-4.16 ct
sept. minor third	$7/6$	1	0	δ_4	+4.75 ct
minor third	$6/5$	2	-2	$\gamma_4^{\uparrow \uparrow}$	+0.59 ct
major third	$5/4$	1	1	$\epsilon_4^\downarrow$	-1.18 ct
sept. major third	$9/7$	2	-1	$\delta_4^\uparrow$	-5.34 ct
fourth	$4/3$	1	2	$\alpha_5^\downarrow$	+0.59 ct
augmented fourth	$7/5$	3	-2	$\delta_4^{\uparrow \uparrow}$	+5.34 ct
diminished fifth	$10/7$	1	3	$\beta_5^{\downarrow \downarrow}$	-5.34 ct
fifth	$3/2$	3	-1	$\epsilon_4^\uparrow$	-0.59 ct
sept. minor sixth	$14/9$	2	2	$\beta_5^\downarrow$	+5.34 ct
minor sixth	$8/5$	3	0	$\alpha_5^\uparrow$	+1.18 ct
major sixth	$5/3$	2	3	$\gamma_5^{\downarrow \downarrow}$	-0.59 ct
sept. major sixth	$12/7$	3	1	β_5	-4.75 ct
sept. minor seventh	$7/4$	4	-1	$\alpha_5^{\uparrow \uparrow}$	+4.16 ct
narrow minor seventh	$16/9$	2	4	$\delta_5^{\downarrow 3}$	+1.18 ct
wide minor seventh	$9/5$	5	-3	$\epsilon_4^{\uparrow 3}$	± 0.00 ct
major seventh	$15/8$	4	0	$\beta_5^\uparrow$	-1.78 ct
octave	$2/1$	4	1	γ_5	± 0.00 ct

w , s , and c have been chosen to minimize the maximum absolute error in cents over all intervals listed in Table 1 (the absolute error is always smaller than 5.35 ct) while ensuring that the octave is perfect (i. e. $w^4 s = 2$).

As can be seen in Figure 4, the intervals do not lie close to a line on the hexagonal keyboard but rather cover a two-dimensional area with some holes.

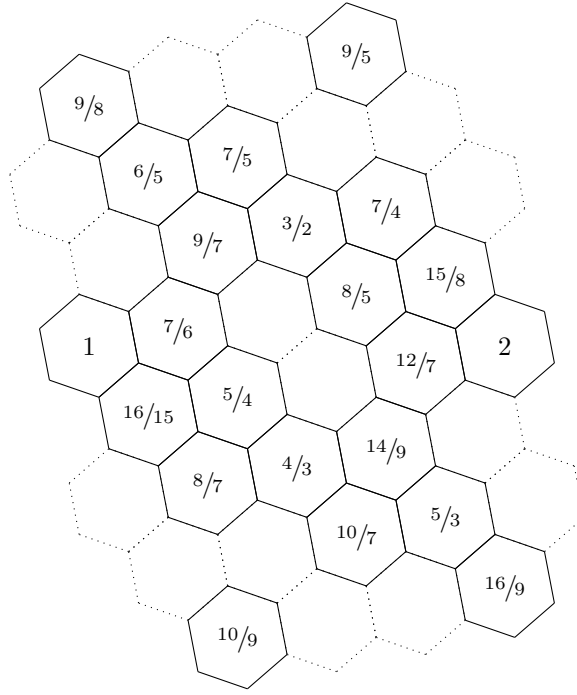


Figure 4: Intervals from Table 1 on the isomorphic keyboard

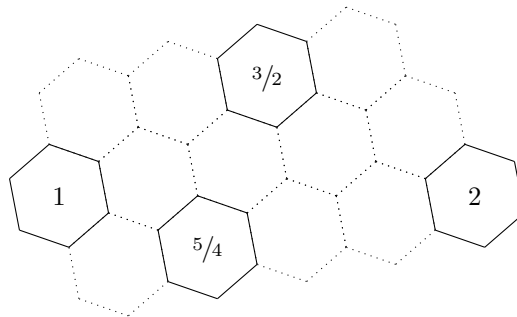


Figure 5: A chord consisting of 4 notes on the isomorphic keyboard, constructed by subdividing the octave (2) into a fifth ($\frac{3}{2}$) and a fourth ($2 : \frac{3}{2} = \frac{4}{3}$) and subdividing the fifth into a major third ($\frac{5}{4}$) and a minor third ($\frac{3}{2} : \frac{5}{4} = \frac{6}{5}$)

As the octaves are perfectly tuned ($w^4s = 2$) and given the extent of the interval coverage in Table 1, any ratio p/q with $1 \leq p \leq 10$ and $1 \leq q \leq 10$ can be reached within a margin of error of 5.35 ct.

2.2 Chords

Chords can be constructed by subdividing an interval into smaller intervals (see Figure 5). For those intervals from Table 1 where the numerator and the denominator of the corresponding just interval is small (≤ 10), this is possible in the following fashion:

- An octave, $2/1$, can be subdivided into
 - $10/9 \times 9/5$, a narrow major second and a wide minor seventh
 - $8/7 \times 7/4$, a sept. major second and a sept. minor seventh
 - $6/5 \times 5/3$, a minor third and a major sixth
 - $5/4 \times 8/5$, a major third and a minor sixth
 - $4/3 \times 3/2$, a fourth and a fifth
 - $7/5 \times 10/7$, an augmented fourth and diminished fifth
- A wide minor seventh, $9/5$, can be subdivided into
 - $9/7 \times 7/5$, a sept. major third and an augmented fourth
 - $9/8 \times 8/5$, a wide major second and a minor sixth
 - $6/5 \times 3/2$, a minor third and a fifth
- A sept. minor seventh, $7/4$, can be subdivided into
 - $7/6 \times 3/2$, a sept. minor third and a fifth
 - $7/5 \times 5/4$, an augmented fourth and a major third
- A major sixth, $5/3$, can be subdivided into
 - $10/9 \times 3/2$, a narrow major second and a fifth
 - $7/6 \times 10/7$, a sept. minor third and a diminished fifth
 - $5/4 \times 4/3$, a major third and a fourth
- A minor sixth, $8/5$, can be subdivided into
 - $8/7 \times 7/5$, a sept. major second and an augmented fourth
 - $6/5 \times 4/3$, a minor third and a fourth
 - $5/4 \times 9/7$ (approx.), a major third and sept. major third
- A fifth, $3/2$, can be subdivided into
 - $9/8 \times 4/3$, a wide major second and a fourth
 - $7/6 \times 9/7$, a sept. minor third and a sept. major third
 - $6/5 \times 5/4$, a minor third and a major third

- A diminished fifth, $^{10}/7$, can be subdivided into
 - $^{10}/9 \times ^9/7$, a narrow major second and a sept. major third
 - $^8/7 \times ^5/4$, a sept. major second and a major third
- An augmented fourth, $^7/5$, can be subdivided into
 - $^9/8 \times ^5/4$ (approx.), a wide major second and a major third
 - $^7/6 \times ^6/5$, a sept. minor third and a minor third
- A fourth, $^4/3$, can be subdivided into
 - $^{10}/9 \times ^6/5$, a narrow major second and a minor third
 - $^8/7 \times ^7/6$, a sept. major second and a sept. minor third
- A sept. major third, $^9/7$, can be subdivided into
 - $^9/8 \times ^8/7$, a wide major second and a sept. major second
- A major third, $^5/4$, can be subdivided into
 - $^{10}/9 \times ^9/8$, a narrow major second and a wide major second

Note that even if a combination of certain just intonation intervals would only result in an approximation ($^{5}/4 \times ^9/7 \approx ^8/5$ and $^9/8 \times ^5/4 \approx ^7/5$, both with an error greater than 5.35 ct), combining the *approximated* intervals on the two-dimensional grid will keep the errors within the 5.35 ct error bound shown in subsection 2.1. The approximation errors compensate for each other, thus allowing for a greater variety of combinations and subdivisions of the intervals in Table 1 within the given error bound compared to just intonation.