Approximation of distance between locations on earth given by latitude and longitude

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In this paper we shall provide a method to approximate distances between two points on earth given by latitude and longitude. The motivation behind this approximation is the efficient mass-calculation of distances between a reference point and other nearby points in a huge database, while saving the expenses of trigonometric calculations for each point in the database. The approximation is done by approximating the squared distance with a 2nd order Taylor-polynomial. Finally we will present a method to calculate latitude and longitude boundaries for a given center point and radius, in order to allow quick lookup of entries using a 2-dimensional database index.

We model the earth using a reference ellipsoid as defined by the formulas below, where a is the semi-major and b is the semi-minor axis of a reference ellipsoid (e.g. WGS-84) in meter.

$$
\begin{aligned}
X &= N_\phi \cos\phi \cos\lambda \\
Y &= N_\phi \cos\phi \sin\lambda \\
Z &= N_\phi (1-\epsilon^2) \sin\phi \\
\epsilon &= \frac{\sqrt{a^2-b^2}}{a} \\
N_\phi &= \frac{a}{\sqrt{1-\epsilon^2\sin^2\phi}}
\end{aligned}
$$

The distance d of two points (X, Y, Z) and (X_0, Y_0, Z_0) in 3-dimensional space is given by:

$$\begin{aligned} d^2 &= (X - X_0)^2 + (Y - Y_0)^2 + (Z - Z_0)^2 \\ &= (N_\phi \cos\phi \cos\lambda - N_{\phi_0} \cos\phi_0 \cos\lambda_0)^2 \\ &\quad + (N_\phi \cos\phi \sin\lambda - N_{\phi_0} \cos\phi_0 \sin\lambda_0)^2 \\ &\quad + (N_\phi (1 - \epsilon^2) \sin\phi - N_{\phi_0} (1 - \epsilon^2) \sin\phi_0)^2 \end{aligned}$$

Note that the path on the surface of the ellipsoid is distinct from d. For our approximation of d^2 with a 2nd order Taylor-polynomial this discrepancy has no negative impact, as it does not affect any of its coefficients (not proven here). The Taylor-polynomial $\tilde{d^2}$ is given by:

$$d^2 \approx \tilde{d^2} = \sum_{i=0}^{2} \sum_{j=0}^{2} \underbrace{\frac{\left(\frac{\partial}{\partial\phi}^i \frac{\partial}{\partial\lambda}^j d^2\right) |_{\phi=\phi_0,\lambda=\lambda_0}}{i!\, j!}}_{=:t_{ij}} (\phi - \phi_0)^i (\lambda - \lambda_0)^j$$

Using a computer algrabra software, we can calcuate t_{ij} and get the following results:

$$\begin{aligned} t_{00} &= 0 \\ t_{10} &= 0 \\ t_{20} &= a^2 \frac{(1 - \epsilon^2)^2}{\left(1 - \epsilon^2 \sin^2\phi_0\right)^3} \\ t_{01} &= 0 \\ t_{11} &= 0 \\ t_{21} &= 0 \\ t_{02} &= a^2 \frac{1 - \sin^2\phi_0}{1 - \epsilon^2 \sin^2\phi_0} \\ t_{12} &= -a^2 \frac{(1 - \epsilon^2) \sin\phi_0 \cos\phi_0}{\left(1 - \epsilon^2 \sin^2\phi_0\right)^2} \\ t_{22} &= -a^2 \frac{(1 - \epsilon^2)\left(1 - \sin^2\phi_0\right)\left(\frac{1}{2} + \epsilon^2 \sin^2\phi_0\right)}{\left(1 - \epsilon^2 \sin^2\phi_0\right)^3} \end{aligned}$$

With $t_{00} = t_{10} = t_{01} = t_{11} = t_{21} = 0$ we can re-write $\tilde{d}^2$ as:

$$
\begin{aligned}
\tilde{d}^2 &= t_{20}(\phi - \phi_0)^2 + (t_{22}(\phi - \phi_0)^2 + t_{12}(\phi - \phi_0) + t_{02})(\lambda - \lambda_0)^2 \\
&= t_{20}\left[(\phi - \phi_0)^2 + \left(\frac{t_{22}}{t_{20}}(\phi - \phi_0)^2 + \frac{t_{12}}{t_{20}}(\phi - \phi_0) + \frac{t_{02}}{t_{20}}\right)(\lambda - \lambda_0)^2\right] \\
&= t_{20}\left[(\phi - \phi_0)^2 + \left(\frac{t_{22}}{t_{20}}\phi^2 - 2\frac{t_{22}}{t_{20}}\phi\phi_0 + \frac{t_{22}}{t_{20}}\phi_0^2 + \frac{t_{12}}{t_{20}}\phi - \frac{t_{12}}{t_{20}}\phi_0 + \frac{t_{02}}{t_{20}}\right)(\lambda - \lambda_0)^2\right] \\
&= t_{20}\left[(\phi - \phi_0)^2 + \left(\frac{t_{22}}{t_{20}}\phi^2 + (\frac{t_{12}}{t_{20}} - 2\frac{t_{22}}{t_{20}}\phi_0)\phi + \frac{t_{22}}{t_{20}}\phi_0^2 - \frac{t_{12}}{t_{20}}\phi_0 + \frac{t_{02}}{t_{20}}\right)(\lambda - \lambda_0)^2\right] \\
&= \underbrace{t_{20}}_{=:c_3}\left[(\phi - \phi_0)^2 + \left(\underbrace{\frac{t_{22}}{t_{20}}}_{=:c_2}\phi^2 + \underbrace{\frac{t_{12} - 2t_{22}\phi_0}{t_{20}}}_{=:c_1}\phi + \underbrace{\frac{t_{22}\phi_0^2 - t_{12}\phi_0 + t_{02}}{t_{20}}}_{=:c_0}\right)(\lambda - \lambda_0)^2\right]
\end{aligned}
$$

Defining 4 constants

$$
\begin{aligned}
c_3 &:= t_{20} \\
c_2 &:= \frac{t_{22}}{t_{20}} \\
c_1 &:= \frac{t_{12} - 2t_{22}\phi_0}{t_{20}} \\
c_0 &:= \frac{t_{22}\phi_0^2 - t_{12}\phi_0 + t_{02}}{t_{20}}
\end{aligned}
$$

we can further simplify $\tilde{d}^2$ to:

$$
\begin{aligned}
d^2 \approx \tilde{d}^2 &= c_3\left[(\phi - \phi_0)^2 + (c_2\phi^2 + c_1\phi + c_0)(\lambda - \lambda_0)^2\right] \\
&= c_3\left[(\phi - \phi_0)^2 + ((c_2\phi + c_1)\phi + c_0)(\lambda - \lambda_0)^2\right]
\end{aligned}
$$

Given a minimum (r) and maximum (R) search radius, we can filter database entries using the following formula:

$$\boxed{\frac{r^2}{c_3} \leq (\phi - \phi_0)^2 + ((c_2\phi + c_1)\phi + c_0)(\lambda - \lambda_0)^2 < \frac{R^2}{c_3}}$$

If latitudes and longitudes are not given as radians but as degrees, we can use alternative coefficients:

$$\begin{aligned}
c_3' &= \left(\frac{\pi}{180}\right)^2 c_3 \\
c_2' &= \left(\frac{\pi}{180}\right)^2 c_2 \\
c_1' &= \frac{\pi}{180} c_1 \\
c_0' &= c_0
\end{aligned}$$

In order to get the latitude boundaries $\phi_b = \phi_0 \pm \Delta\phi_b$ for a given center point (ϕ_0, λ_0) and maximum search radius (R), we use the simplified equation for $\tilde{d}^2$ and set $\tilde{d}^2 = R^2$, $\phi = \phi_b$ and $\lambda = \lambda_0$:

$$\begin{aligned}
& \tilde{d}^2 = c_3 \left[(\phi - \phi_0)^2 + (c_2\phi^2 + c_1\phi + c_0)(\lambda - \lambda_0)^2\right] \\
\Rightarrow \quad & R^2 = c_3(\phi_b - \phi_0)^2 + c_3(c_2\phi_b^2 + c_1\phi_b + c_0)(\lambda_0 - \lambda_0)^2 \\
\Leftrightarrow \quad & R^2 = c_3(\phi_b - \phi_0)^2 \\
\Leftrightarrow \quad & (\phi_b - \phi_0)^2 = \frac{R^2}{c_3} \\
\Leftrightarrow \quad & \Delta\phi_b = |\phi_b - \phi_0| = \sqrt{\frac{R^2}{c_3}}
\end{aligned}$$

The longitude boundaries $\lambda_\mathrm{b} = \lambda_0 \pm \Delta\lambda_\mathrm{b}$ are calculated by setting $\tilde{d}^2 = R^2$, $\lambda = \lambda_\mathrm{b}$ and ϕ to a critical value ϕ_c, which maximizes $|\lambda_\mathrm{b} - \lambda_0|$:

$$
\begin{aligned}
& \tilde{d}^2 = c_3 \left[(\phi - \phi_0)^2 + (c_2\phi^2 + c_1\phi + c_0)(\lambda - \lambda_0)^2\right] \\
\Rightarrow \quad & R^2 = c_3 \left[(\phi_\mathrm{c} - \phi_0)^2 + (c_2\phi_\mathrm{c}^2 + c_1\phi_\mathrm{c} + c_0)(\lambda_\mathrm{b} - \lambda_0)^2\right] \\
\Leftrightarrow \quad & (\phi_\mathrm{c} - \phi_0)^2 + (c_2\phi_\mathrm{c}^2 + c_1\phi_\mathrm{c} + c_0)(\lambda_\mathrm{b} - \lambda_0)^2 = \frac{R^2}{c_3} \\
\Leftrightarrow \quad & (c_2\phi_\mathrm{c}^2 + c_1\phi_\mathrm{c} + c_0)(\lambda_\mathrm{b} - \lambda_0)^2 = \frac{R^2}{c_3} - (\phi_\mathrm{c} - \phi_0)^2 \\
\Leftrightarrow \quad & (\lambda_\mathrm{b} - \lambda_0)^2 = \frac{\frac{R^2}{c_3} - (\phi_\mathrm{c} - \phi_0)^2}{c_2\phi_\mathrm{c}^2 + c_1\phi_\mathrm{c} + c_0} \\
\Leftrightarrow \quad & \Delta\lambda_\mathrm{b} = |\lambda_\mathrm{b} - \lambda_0| = \sqrt{\frac{\frac{R^2}{c_3} - (\phi_\mathrm{c} - \phi_0)^2}{c_2\phi_\mathrm{c}^2 + c_1\phi_\mathrm{c} + c_0}}
\end{aligned}
$$

ϕ_c can be determined by setting $\frac{\partial((\lambda_\mathrm{b} - \lambda_0)^2)}{\partial\phi_\mathrm{c}} = 0$. Using a computer algebra software, we get two results, of which only the following result leads to $\phi_\mathrm{c} \in [-\frac{\pi}{2}; +\frac{\pi}{2}]$ and real λ_b:

$$
\phi_\mathrm{c} = \frac{c_2(\phi_0^2 - \frac{R^2}{c_3}) - c_0 + \sqrt{\begin{array}{c} c_2^2\ \phi_0^4\ +\ 2c_1c_2\ \phi_0^3\ +\ (c_1^2 + 2(c_0 - c_2\frac{R^2}{c_3})c_2)\ \phi_0^2 \\ +\ 2(c_0 - c_2\frac{R^2}{c_3})c_1\ \phi_0 \\ +\ c_2^2\left(\frac{R^2}{c_3}\right)^2 + (2c_0c_2 - c_1^2)\ \frac{R^2}{c_3} + c_0^2 \end{array}}}{2c_2\phi_0 + c_1}
$$

The above formula only holds for cases where $\phi_0 \neq 0$ and the north or south pole is not included within the search radius. If $\phi_0 = 0$, then $\phi_\mathrm{c} = 0$. It is recommended to limit ϕ_c to an absolute value (ϕ_limit), which is the maximum possible absolute value of ϕ in practice, e.g. $84^\circ = \frac{7}{15}\pi$. When $\phi_\mathrm{c} > \phi_\mathrm{limit}$, then ϕ_limit should be used instead of ϕ_c to calculate $\Delta\lambda_\mathrm{b}$. When $\phi_\mathrm{c} < -\phi_\mathrm{limit}$, then $-\phi_\mathrm{limit}$ should be used respectivly. When $\phi_\mathrm{c} \notin \mathbb{R}$, because a pole is included within the search radius, then ϕ_limit can be used for cases where $\phi_0 > 0$, and $-\phi_\mathrm{limit}$ can be used for cases where $\phi_0 < 0$.

In addition to the outer boundaries $\phi_\mathrm{b} = \phi_0 \pm \Delta\phi_\mathrm{b}$ and $\lambda_\mathrm{b} = \lambda_0 \pm \Delta\lambda_\mathrm{b}$ for the maximum radius R, it is possible to define inner boundaries $(\phi_\mathrm{i}, \lambda_\mathrm{i})$ for the minimum radius r. There are infinite possible solutions, we pick one by chosing λ_i as follows:

$$\Delta\lambda_\mathrm{i} = |\lambda_\mathrm{i} - \lambda_0| = \sqrt{\frac{\frac{r^2}{c_3}}{2\left(c_2\phi_0^2 + c_1\phi_0 + c_0\right)}}$$

Our choice of λ_i is optimal for $\phi_0 = 0$ and $r \to 0$ (not proven here). Having chosen λ_i, we can calculate ϕ_i by using the simplified equation for $\tilde{d}^2$ and setting $\tilde{d}^2 = r^2$, $\phi = \phi_\mathrm{i}$ and $\lambda = \lambda_\mathrm{i}$:

$$\begin{aligned}
& \tilde{d}^2 = c_3\left[(\phi - \phi_0)^2 + (c_2\phi^2 + c_1\phi + c_0)(\lambda - \lambda_0)^2\right] \\
\Rightarrow\quad & r^2 = c_3\left[(\phi_\mathrm{i} - \phi_0)^2 + (c_2\phi_\mathrm{i}^2 + c_1\phi_\mathrm{i} + c_0)(\lambda_\mathrm{i} - \lambda_0)^2\right] \\
\Leftrightarrow\quad & r^2 = c_3\left[(\phi_\mathrm{i} - \phi_0)^2 + (c_2\phi_\mathrm{i}^2 + c_1\phi_\mathrm{i} + c_0)(\Delta\lambda_\mathrm{i})^2\right] \\
\Leftrightarrow\quad & \frac{r^2}{c_3} = (\phi_\mathrm{i} - \phi_0)^2 + (c_2\phi_\mathrm{i}^2 + c_1\phi_\mathrm{i} + c_0)(\Delta\lambda_\mathrm{i})^2 \\
\Leftrightarrow\quad & \frac{r^2}{c_3} = \phi_\mathrm{i}^2 - 2\phi_\mathrm{i}\phi_0 + \phi_0^2 + c_2(\Delta\lambda_\mathrm{i})^2\phi_\mathrm{i}^2 + c_1(\Delta\lambda_\mathrm{i})^2\phi_\mathrm{i} + c_0(\Delta\lambda_\mathrm{i})^2 \\
\Leftrightarrow\quad & \left(c_2(\Delta\lambda_\mathrm{i})^2 + 1\right)\phi_\mathrm{i}^2 + \left(c_1(\Delta\lambda_\mathrm{i})^2 - 2\phi_0\right)\phi_\mathrm{i} + c_0(\Delta\lambda_\mathrm{i})^2 + \phi_0^2 - \frac{r^2}{c_3} = 0 \\
\Leftrightarrow\quad & \phi_i = \frac{\phi_0 - \frac{1}{2}c_1(\Delta\lambda_\mathrm{i})^2 \pm \sqrt{\begin{array}{c}(\frac{1}{4}c_1^2 - c_0c_2)(\Delta\lambda_\mathrm{i})^4 \\ +(c_2\frac{r^2}{c_3} - (c_2\phi_0^2 + c_1\phi_0 + c_0))(\Delta\lambda_\mathrm{i})^2 \\ +\frac{r^2}{c_3}\end{array}}}{c_2(\Delta\lambda_\mathrm{i})^2 + 1}
\end{aligned}$$

$$\Leftrightarrow\quad \phi_i = \frac{\phi_0 - \frac{1}{2}c_1(\Delta\lambda_\mathrm{i})^2 \pm \sqrt{(\frac{1}{4}c_1^2 - c_0c_2)(\Delta\lambda_\mathrm{i})^4 + c_2\frac{r^2}{c_3}(\Delta\lambda_\mathrm{i})^2 + \frac{1}{2}\frac{r^2}{c_3}}}{c_2(\Delta\lambda_\mathrm{i})^2 + 1}$$

Note that the two solutions of ϕ_i are normally not symmetrical to ϕ_0.